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Maslov class and minimality in Calabi–Yau manifolds

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Abstract

Generalizing the construction of the Maslov class $[\mu_\Lambda]$ for a Lagrangian embedding in a symplectic vector space, we prove that it is possible to give a consistent definition of the class $[\mu_\Lambda]$ for any Lagrangian submanifold of a Calabi–Yau manifold. Moreover, extending a result of Morvan in symplectic vector spaces, we prove that $[\mu_\Lambda]$ can be represented by $i_H\omega$, where H is the mean curvature vector field of the Lagrangian embedding and ω is the Kähler form associated to the Calabi–Yau metric. Finally, we conjecture a generalization of the Maslov class for Lagrangian submanifolds of any symplectic manifold via the mean curvature representation.

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1. Introduction

The Maslov class $[\mu_\Lambda]$ of a Lagrangian embedding $j : \Lambda \hookrightarrow V$ in the standard Euclidean symplectic vector space V has been constructed by Maslov in the study of global patching problem for asymptotic solutions of some PDEs (see [13] for further details on this point of view). Subsequently, this cohomological class has found applications in the analysis of several quantization procedures, starting from [1] up to recent aspects on its relations with asymptotic, semiclassical and geometric quantization, for which we refer to [9,11]. In spite of this, there are several problems in the very definition of the Maslov class for Lagrangian submanifolds of generic symplectic manifolds.

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In [14] it has been proved that, for a Lagrangian embedding $j : \Lambda \hookrightarrow V$ in a Euclidean symplectic vector space (V, ω), the Maslov form μ_Λ can be represented by $\mu_\Lambda = i_H\omega$, that is by the contraction of the symplectic form with the mean curvature vector field H of the embedding j. Unfortunately, the very definition of Maslov form (and related class) as exposed in [1,2,13] depends on the fact that the Lagrangian submanifold Λ is embedded in a symplectic vector space, in which we have chosen a projection $\pi : V \to \Lambda_0$ over a *fixed* Lagrangian subspace Λ_0; then the Maslov class $[\mu_\Lambda] \in H^1(\Lambda, \mathbb{R})$ can be defined as the Poincaré dual to the singular locus $Z(\Lambda) \hookrightarrow \Lambda$, where $Z(\Lambda) := \{\Lambda \in \Lambda | \mathrm{rk}(\pi_*(\Lambda)) < \max\} \cap H_{n-1}(\Lambda, \mathbb{Z})$. In the classical literature it is proved that if one changes projection π, that is if one changes the reference Lagrangian subspace Λ_0, then the *Maslov class* μ_Λ does not change, while its representative changes. This is achieved using the so-called universal Maslov class construction on the Lagrangian Grassmannian $\mathrm{GrL}(V)$ (the homogeneous space which parametrizes Lagrangian subspaces of (V, ω), see [1,2,11]). These formulations depend heavily on the linear structure of the ambient manifold V; in particular it is assumed that V is endowed with the trivial connection. Therefore, it seems difficult even to define the Maslov class for Lagrangian submanifolds of symplectic manifolds, which are not vector spaces. For instance, it is possible to define the Maslov class of a Lagrangian embedding via the so-called generating functions, or their generalization (Morse families), for which we refer to [13], and particularly [18]. In this way, one obtains a notion of Maslov class for Lagrangian submanifolds embedded in *any* cotangent bundle T^*M over a Riemannian manifold M, constructing a $\mathbb{Z}$-valued Čech cocycle, starting from the signature of the Hessian of a Morse family; however this construction depends strongly on the choice of a “base manifold” (M in the case of the cotangent bundle) and does not seem to be generalizable to Lagrangian embedding in any symplectic manifold (see [18] for more details on this kind of construction).

Recently, Fukaya [7] has shown how to define a Maslov index for closed loops on Lagrangian submanifolds of a quite general class of symplectic manifolds, the so-called pseudo-Einstein symplectic manifolds. The construction is developed using nontrivial assumptions on the structure of the ambient manifold and is carried on only for a particular subclass of Lagrangian submanifolds; moreover, there is no explicit reference to the corresponding Maslov class.

In this paper we show that, whenever the ambient manifold is Calabi–Yau, it is possible to give a consistent definition of Maslov class for its Lagrangian submanifolds, generalizing the approach of Arnol’d with the so-called universal Maslov class. In this framework, we show that it is possible to generalize the result of Morvan and then we comment on various consequences of our construction, in particular on the possible definition of Maslov class for Lagrangian embedding in any symplectic manifold.

2. The Maslov class for Lagrangian embedding in Calabi–Yau

Let us briefly recall the standard construction of the Maslov class μ_Λ, for a Lagrangian submanifold Λ, embedded in a symplectic vector space (V, ω), of real dimension $2n$: first of all, one considers the tangent spaces to Λ as (affine) subspaces of V. Then, using the trivial parallel displacement one transports every tangent plane in a fixed point P of V (for example the origin). Now, one has to consider the Lagrangian Grassmannian $\mathrm{GrL}(T_P V)$, which by definition parametrizes all Lagrangian subspaces of $T_P V$. Using the trivial connection, we have thus obtained a map

$$G : \Lambda \to \mathrm{GrL}(T_P V).$$

It is easy to see [1,2] that $\mathrm{GrL}(T_P V)$ has the natural structure of the homogeneous space $U(n)/O(n)$; then by the standard tool of the exact homotopy sequence for a fibration (see [6]), it is proved that $\pi_1(\mathrm{GrL}(T_P V)) \cong \mathbb{Z}$. In fact, having fixed a Lagrangian plane Λ_0 in $T_P V$, all other Lagrangian planes are obtained via a unitary automorphism $A \in U(n)$. Obviously, we have a fibration

$$\mathrm{SU}(n) \to U(n) \xrightarrow{\det} S^1,$$

but this does not descend to $\mathrm{GrL}(T_P S)$, since we have to quotient out the possible orthogonal automorphisms. However, since the square of the determinant of an orthogonal automorphism is always 1, we have a well-defined map

$$\det{}^2 : \mathrm{GrL}(T_P S) \to S^1,$$

which sits in the following commutative diagram of fibrations:

$$\begin{array}{ccccc}
\mathrm{SO}(n) & \to & O(n) & \xrightarrow{\det} & S^0 \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{SU}(n) & \to & U(n) & \xrightarrow{\det} & S^1 \\
\downarrow & & \downarrow & & \downarrow z^2 \\
\mathrm{GrSL}(\mathbb{C}^n) & \to & \mathrm{GrL}(\mathbb{C}^n) & \xrightarrow{\det^2} & S^1
\end{array}$$

In this diagram the space $\mathrm{GrSL}(\mathbb{C}^n)$ denotes the Grassmannian of special Lagrangian planes in $\mathbb{C}^n$, that is the Grassmannian of Lagrangian planes which are *calibrated* by the top holomorphic form of $\mathbb{C}^n$; the corresponding Lagrangian submanifolds are called special Lagrangian (see [10] for more details). Notice that this space is always simply connected.

Finally, using Hurewicz isomorphism and taking a generator belonging to $H^1(\mathrm{GrL}(T_P V), \mathbb{Z})$, which is thought as the pull-back via $\det^2$ of the generator $[\alpha] \in H^1(S^1, \mathbb{Z})$, one defines the Maslov class $[\mu_\Lambda] := G^*(\det^2)^*[\alpha]$. Obviously, this construction is independent on the choice of the point P, since if another point is chosen it is possible to construct a homotopy in such a way so as to prove the invariance of $[\mu_\Lambda]$. It is clear that, in this framework, the existence of the trivial connection is an (almost!) essential requirement for the construction to work. In fact, we will see in this section that to have a consistent definition of Maslov class it is not necessary that the ambient manifold is endowed with the trivial connection, but is sufficient that the global holonomy of the symplectic manifold is “small” in a suitable sense.

From now on we restrict our attention to Lagrangian submanifolds of Calabi–Yau manifolds. Recall that Calabi–Yau manifolds can be defined as compact Kähler manifolds with vanishing first Chern class; recall also that a celebrated theorem by Yau (proving a previous conjecture by Calabi) implies that for every choice of the Kähler class on a Calabi–Yau, there exists a unique Ricci-flat Kähler metric. Moreover, while the holonomy of a Kähler manifold is contained in $U(n)$, if g is the Ricci-flat metric of an n-dimensional Calabi–Yau, then the corresponding holonomy group is contained in $\mathrm{SU}(n)$. Finally, let us recall that, on every Kähler manifold (X, g, J) (where g is a Kähler metric and J the integrable almost complex structure) the corresponding symplect or Kähler form ω is related to g via

$$\omega(X, Y) := g(X, JY) \quad \forall X, Y \in \Gamma(TX) \tag{1}$$

and that the almost complex structure tensor J is covariantly constant with respect to the Levi-Civita connection induced by g. Considering a Kähler metric g on a Calabi–Yau, we will always mean the Ricci-flat metric. Typical examples of Calabi–Yau are given by the zero locus of a homogeneous polynomial of degree $n+1$ in $\mathbb{P}^n(\mathbb{C})$ (whenever this locus is smooth); however, it is by no means true that all Calabi–Yau are algebraic. For further details on this class of manifolds see for example [4,17].

The construction of Fukaya for defining the Maslov index of closed loops goes as follows (see [7] for details and motivations). He considers symplectic manifolds (X, ω) which are "pseudo-Einstein" in the sense that there exists an integer N such that $N\omega = c_1(X)$. By this relation, the line bundle $\det(TX)$ is flat when restricted to every Lagrangian submanifold Λ of X, but Fukaya restricts further the class of Lagrangian submanifolds considering only the so-called Bohr–Sommerfeld orbit Λ (BS-orbit for short), which are defined as the Lagrangian submanifolds for which the restriction of $\det(TX)$ is not only flat, but even trivial. This implies that if we consider a closed loop $h : S^1 \to \Lambda$ (Λ is a BS-orbit), then the monodromy M of the tangent bundle TX along $h(S^1)$ is contained in $\mathrm{SU}(n)$. Then the idea is to take a path in $\mathrm{SU}(n)$ joining M with the identity in order to get an induced trivialization of $h^*(TX_{|h(S^1)}) \cong S^1 \times \mathbb{C}^n$. In this trivial bundle there is a family of Lagrangian vector subspaces $T_{h(t)}\Lambda$ and in this way we get a loop in $\mathrm{GrL}(\mathbb{C}^n)$ and hence a well-defined integer (the Maslov index) $m(h)$. Obviously $m(h)$ is independent of the choice of the path in $\mathrm{SU}(n)$ which joins M to the unit, since $\pi_1(\mathrm{SU}(n)) \cong 1$.

Now we come to our construction. Consider embedded Lagrangian submanifolds Λ of a Calabi–Yau (X, ω, g, J), where ω, g, J are related by (1). Define the *Lagrangian Grassmannization* $\mathrm{GrL}(X)$ of TX as the fibre bundle over X obtained by substituting T_xX with $\mathrm{GrL}(T_xX)$, thus

$$\mathrm{GrL}(X) := \coprod_{x\in X} \mathrm{GrL}(T_xX)$$

and in particular

$$\mathrm{GrL}(X)_\Lambda := \coprod_{x\in \Lambda} \mathrm{GrL}(T_xX).$$

Let $G(j)$ be the Gauss map, which takes $x \in \Lambda$ in $T_x\Lambda$ thought as a Lagrangian subspace of T_xX. Via $G(j)$, the embedding $j : \Lambda \hookrightarrow X$ lifts to a section $G(j) : \Lambda \to \mathrm{GrL}(X)_\Lambda$. We would like to define the Maslov class of Λ via a map $\mathcal{M} : \Lambda \to S^1$ in the following way: to every point $x \in \Lambda$, we consider $G(j)(x)$ and then through the isomorphism $\mathrm{GrL}(T_xX) \cong$

$U(n)/O(n)$, taking the map $\det^2$ we get a point in S^1. However, as we have seen, to establish an isomorphism to every space $\mathrm{GrL}(T_xX)$ $(x \in \Lambda)$ with $U(n)/O(n)$ we need a reference Lagrangian plane in $\mathrm{GrL}(T_xX) \forall x \in \Lambda$, that is we need *another section* of $\mathrm{GrL}(X)_\Lambda$, besides $G(j)(\Lambda)$.

To this aim, fix a point $p \in \Lambda$, consider $T_p\Lambda$ and use the parallel displacement, induced by the Levi-Civita connection of g, along a system γ of paths on Λ starting from p, to construct a reference distribution of Lagrangian planes $\mathcal{D}_\gamma$ over Λ that is another section of $\mathrm{GrL}(X)_\Lambda$. This is indeed possible, since the holonomy is contained in $U(n)$, the parallel displacement is an isometry for g and J is covariantly constant: these facts combined with the relation (1) imply that parallel transport sends Lagrangian planes in Lagrangian planes. Obviously this distribution $\mathcal{D}_\gamma$ is not uniquely determined, since it depends on the choice of the system of paths γ starting from p. In spite of this, due to the fact that the holonomy of a Calabi–Yau metric is very constrained, this dependence does not prevent us to reach our goal. Indeed, consider $q \in \Lambda$ and compare the two Lagrangian planes $(\mathcal{D}_\gamma)_q$ and $(\mathcal{D}_\delta)_q$ obtained by parallel transport of $T_p\Lambda$ along two different paths γ and δ. By the holonomy property of a Calabi–Yau metric we have

$$(\mathcal{D}_\gamma)_q = M(\mathcal{D}_\delta)_q, \quad M \in \mathrm{SU}(n).$$

Thus, if $A \in U(n)$ is such that $T_q\Lambda = A(\mathcal{D}_\gamma)_q$, then $T_q\Lambda = AM(\mathcal{D}_\delta)_q$; so to every $q \in \Lambda$ we can associate A_q such that $G(j)(q) = T_q\Lambda = A_q(\mathcal{D}_\gamma)_q$, where A_q is determined up to multiplication by a matrix $M \in \mathrm{SU}(n)$. At this point the key observation is that $\det^2(A_q) \in S^1$ is a well-defined point, which is not affected by the ambiguity of A_q. In this way we have a well-defined map, the *Maslov map*

$$\begin{array}{cccc} \mathcal{M}: & \Lambda & \to & S^1 \\ & q & \mapsto & \det^2(A_q) \end{array}$$

Take the generator $[\alpha]$ of $H^1(S^1, \mathbb{Z})$ represented by the form $\alpha := (1/2\pi)\,\mathrm{d}\theta$. Observe that the target space of the Maslov map is not only topologically a circle, but even a Lie group, the group $U(1)$: this implies that the choice of the form $(1/2\pi)\,\mathrm{d}\theta$ is compulsory, since it is the unique normalized invariant 1-form. Now we can give the following:

Definition. Using the previous notations, we define the *Maslov form*of the Lagrangian embedding $j : \Lambda \hookrightarrow X$ as $\mu_\Lambda := \mathcal{M}^*\alpha$ and the corresponding *Maslov class* as $[\mu_\Lambda] = \mathcal{M}^*[\alpha] \in H^1(\Lambda, \mathbb{Z})$.

Remark 1. *The Maslov map $\mathcal{M}$ has been built up fixing a reference point p, from which we constructed $\mathcal{D}_\gamma$; in this way the map $\mathcal{M}$ associates to p $1 \in S^1$. It is clear that if one takes a different reference point p', then the map $\mathcal{M}$ changes* (*this time p' goes to* 1), *but the Maslov class and the Maslov form do not change, as it is immediate to see. In particular, the invariance of the Maslov form is due to the invariance of α under the action of the Lie group $U(1)$.*

Remark 2. *In* [16], *Trofimov constructed a generalized Maslov class as a cohomological class defined on the space of paths* $[X, \Lambda]$; *these paths start from a fix point* x_0 *in a symplectic manifold X and end to a fixed Lagrangian submanifold* Λ *of X. We argue that the Maslov class we have just defined can be obtained as a finite dimensional reduction of the class built up in* [16], *when one uses the Levi-Civita connection induced by the Calabi–Yau metric. In fact, Trofimov did not use metric connections, but instead affine torsion-free connections, preserving the symplectic structure, which are generally not induced by a metric.*

3. Representation of the Maslov class via the mean curvature vector field

In this section, generalizing what has been proved by Morvan in [14] for Lagrangian embeddings in Euclidean symplectic vector space, we prove the following:

Theorem. *Let* $j : \Lambda \hookrightarrow X$ *be a Lagrangian embedding in a Calabi–Yau X and let* $H \in \Gamma(N\Lambda)$ *be the mean curvature vector field of the embedding j* (*with respect to the Calabi–Yau metric*), *then*

$$\mu_\Lambda = \frac{1}{\pi} i_H \omega,$$

where ω is the Kähler form constructed from the Calabi–Yau metric g, *and* μ_Λ *is the Maslov form previously defined.*

Before proving the theorem we need various preliminary results, which we are going to state and prove, and we also need to decompose into simpler pieces the action of $\mathcal{M}^*$ on $[\alpha]$.

Recall that given an embedding j, the associated second fundamental form $\sigma : T\Lambda \times T\Lambda \to N\Lambda$ is a symmetric tensor defined by

$$\sigma(X, Y) := \nabla^g_X Y - \nabla^{j^*g}_X Y, \quad \forall X, Y \in \Gamma(T\Lambda),$$

where ∇^g is the Levi-Civita connection in the ambient manifold, while ∇^{j^*g} is the connection induced on Λ via the pulled-back metric. If σ is identically vanishing, then the submanifold is called totally geodesic. Taking the trace of σ we get a field of normal vectors, that is the *mean curvature vector field* H of the embedding j. Those embeddings for which H is identically vanishing are called minimal.

First of all we need to understand the local structure of $T\,\mathrm{GrL}(T_x X)$. Fix a point $q \in \Lambda$ and set $V := T_q X$ for short. We can prove the following.

Lemma 1. *The space* $T_\pi \mathrm{GrL}(V)$ *over a Lagrangian n-plane* π *of V can be identified with the subspace of linear maps* $\psi : \pi \to \pi^\perp$ ($\pi^\perp$ *denotes the orthogonal subspace in V with respect to the metric g in q*) *such that*

$$g(\psi(X), JY) = g(\psi(Y), JX), \quad \forall X, Y \in \pi.$$

Proof. First of all, we have $T_\pi \mathrm{GrL}(V) \equiv S(\pi)$, where $S(\pi)$ is the space of all symmetric bilinear forms on π. In fact every $v \in T_\pi \mathrm{GrL}(V)$ can be represented as $(\mathrm{d}/\mathrm{d}t)B(t)\pi_{|t=0}$, where $B(t)$ is a path of linear symplectic transformation of V, with the condition $B(0) = id_V$. To $v \in T_\pi \mathrm{GrL}(V)$ we can associate a form S_v given by

$$S_v(X, Y) := \omega\left(\frac{\mathrm{d}}{\mathrm{d}t} B(t) X_{|t=0}, Y\right).$$

This form is clearly bilinear and is symmetric

$$\begin{aligned} S_v(X, Y) &= \omega\left(\frac{\mathrm{d}}{\mathrm{d}t} B(t) X_{|t=0}, B(t) Y_{|t=0}\right) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\omega(B(t)X, B(t)Y)_{|t=0} - \omega\left(B(t)X_{|t=0}, \frac{\mathrm{d}}{\mathrm{d}t} B(t) Y_{|t=0}\right) \\ &= 0 - \omega\left(X, \frac{\mathrm{d}}{\mathrm{d}t} B(t) Y_{|t=0}\right) = \omega\left(\frac{\mathrm{d}}{\mathrm{d}t} B(t) Y_{|t=0}, X\right) = S_v(Y, X) \end{aligned}$$

by the fact that $B(t)$ is a symplectic linear transformation of V and by skewsymmetry of ω. It is easy to verify that the corresponding map $T_\pi \mathrm{GrL}(V) \to S(\pi)$ is an isomorphism. Moreover, we have

$$S_v(X, Y) = \omega\left(\frac{\mathrm{d}}{\mathrm{d}t} B(t) X_{|t=0}, Y\right) \underset{(1)}{=} g\left(\frac{\mathrm{d}}{\mathrm{d}t} B(t) X_{|t=0}, JY\right),$$

and thus, identifying $\psi : \pi \to \pi^\perp$ with $(\mathrm{d}/\mathrm{d}t)B(t)\pi_{|t=0}$ we get the result. □

By Lemma 1 it is clear that J itself, restricted to q, can be considered not only as an element of $T_\pi \mathrm{GrL}(V)$ but even as an invariant vector field on $\mathrm{GrL}(V)$, that is $J_q \in \Gamma(T\,\mathrm{GrL}(V))$. Let $e_1, \ldots, e_n$ be an orthonormal basis of π and $f^1, \ldots, f^n$ the corresponding dual basis, in such a way that $Je_1, \ldots, Je_n$ is a basis of $\pi^\perp$ and $-Jf^1, \ldots, -Jf^n$ the associated dual basis. Then J as a vector belonging to $T_\pi \mathrm{GrL}(V)$ can be represented as a section of $\pi^* \otimes \pi^\perp$, that is $J = f^i \otimes Je_i$ (Einstein summation convention is intended). From J in this representation one can construct a 1-form $\tilde{J} \in \omega^1(\mathrm{GrL}(V))$ using the pairing induced by the metric, that is $\tilde{J} = e_i \otimes -Jf^i$. This 1-form has quite an outstanding role.

Lemma 2. *Fix an arbitrary Lagrangian plane in V in order to have a map* $\det^2 : \mathrm{GrL}(V) \to S^1$. *Then*

$$(\det^2)^*(\alpha) = \frac{1}{\pi}\tilde{J},$$

so that $\tilde{J}$ *defines a closed form on* $\mathrm{GrL}(V)$.

Proof. It is sufficient to prove that for every $X \in T_\pi \mathrm{GrL}(V)$ one has $(\det^2)^*(\alpha)(X) = \frac{1}{\pi}\tilde{J}(X)$. Indeed

$$(\det^2)^*(\alpha)(X) = (\alpha)(\det^2_*(X)),$$

so we are led to compute the tangent map to $\det^2$. Assume for simplicity that π is the reference Lagrangian plane in the isomorphism $\mathrm{GrL}(V) \cong U(n)/O(n)$, so that it is represented by the identity matrix. Then, since $T_\pi \mathrm{GrL}(V) \cong u(n)/o(n)$, consider a path $\gamma : (-\epsilon, \epsilon) \to u(n)$, such that $\gamma(0) = \mathbb{O}$ and such that its image in $u(n)$ has an empty intersection with $o(n)$ (except for the zero matrix). The exponential mapping determines in this way a path in $\mathrm{GrL}(V)$ through π. Now, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\det{}^2(\mathrm{e}^{\gamma(t)})_{|t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\det(\mathrm{e}^{2\gamma(t)})_{|t=0} = \frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{2Tr(\gamma(t))})_{|t=0} = 2Tr(\dot{\gamma}(0)) = 2Tr(X),$$

where $\dot{\gamma}(0)$ is identified with the tangent vector X in $T_\pi \mathrm{GrL}(V)$. Hence one gets

$$(\det{}^2)^*(\alpha)(X) = (\alpha)(\det{}^2_*(X)) = (\alpha)(2Tr(X)) = \frac{1}{\pi}\mathrm{Tr}(X).$$

On the other hand, $X \in \Gamma(\pi^* \otimes \pi^\perp)$, so that it can be represented as $X = X^l_k f^k \otimes Je_l$; thus one gets

$$\tilde{J}(X) = (e_i \otimes -Jf^i)(X^l_k f^k \otimes Je_l) = X^i_i = Tr(X). \qquad \square$$

Till now we have worked only locally, having fixed a point $q \in \Lambda$. To proceed we need to globalize the properties stated in Lemmas 1 and 2. Let us define the *vertical tangent bundle* $VT(\mathrm{GrL}(X)_\Lambda)$ ($VT(\mathrm{GrL})$ for short) over $\mathrm{GrL}(X)_\Lambda$ as

$$VT(\mathrm{GrL}(X)_\Lambda) := \coprod_{x\in\Lambda} T\,\mathrm{GrL}(T_x X)$$

notice that this is not the tangent bundle of $\mathrm{GrL}(X)_\Lambda$, since it is obtained by taking the tangent bundle of the fibre only (thus the name vertical). Analogously, one can define the *vertical cotangent bundle* over $\mathrm{GrL}(X)_\Lambda$ as

$$VT^*(\mathrm{GrL}(X)_\Lambda) := \coprod_{x\in\Lambda} T^*\mathrm{GrL}(T_x X)$$

(from now on denoted as $VT^*(\mathrm{GrL})$ for short).

Now, by the previous reasoning and since J is covariantly constant on a Kähler manifold X, we have that J defines a section of $VT(\mathrm{GrL})$ and analogously $\tilde{J}$ induces a section of $VT^*(\mathrm{GrL})$. In order to globalize the result of Lemma 2, observe that the section $\mathcal{D}_\gamma$ of $\mathrm{GrL}(X)_\Lambda$ over Λ, defined in the previous section, enables one to give a well-defined map $\mathrm{Det}^2 : \mathrm{GrL}(X)_\Lambda \to S^1$ (one takes as a reference Lagrangian plane in $\mathrm{GrL}(T_x X)$ the subspace $(\mathcal{D}_\gamma)_x$). It is clear that one gets immediately the following.

Corollary 1. *Under the previous notations and considering the fibration* $\mathrm{Det}^2 : \mathrm{GrL}(X)_\Lambda \to S^1$ *induced by the reference distribution* $\mathcal{D}_\gamma$, *one has*

$$(\mathrm{Det}^2)^*(\alpha) = \frac{1}{\pi}\tilde{J},$$

where $\tilde{J}$ *is viewed as a section of* $VT^*(\mathrm{GrL})$.

Via the Gauss map we can pull-back $VT(\mathrm{GrL})$ to Λ:

$$\begin{array}{ccc} G(j)^*(VT(\mathrm{GrL})) & & VT(\mathrm{GrL}) \\ \downarrow & & \downarrow pr_{VT} \\ \Lambda & \rightarrow & \mathrm{GrL}(X)_\Lambda \end{array}$$

Lemma 3. *The bundle $G(j)^*VT(\mathrm{GrL})$ can be identified with the subspace of $T^*\Lambda \otimes N\Lambda$ consisting of those sections $\psi \in \Gamma(T^*\Lambda \otimes N\Lambda)$ (that is $N\Lambda$-valued 1-forms on Λ) such that*

$$g(\psi(X), JY) = g(\psi(Y), JX), \quad \forall X, Y \in \Gamma(T\Lambda).$$

Proof. By the very definition of pulled-back bundle, we have that

$$\begin{aligned} G(j)^*VT(\mathrm{GrL}) \cong \{&(x; x', \pi, X) \in \Lambda \times VT(\mathrm{GrL}) : (x, T_x\Lambda) \\ &= G(j)(x) = pr_{VT}(x', \pi, X) = (x', \pi)\}, \end{aligned}$$

which clearly implies the constraint $x = x'$ and $T_x\Lambda = \pi$ so that

$$G(j)^*VT(\mathrm{GrL}) \cong \coprod_{x\in\Lambda} T_{\pi=T_x\Lambda}\mathrm{GrL}(T_xX).$$

On the other hand, by Lemma 1,

$$\begin{aligned} T_{\pi=T_x\Lambda}\mathrm{GrL}(T_xX) \cong \{&\psi \in \Gamma(T_x^*\Lambda \otimes N_x\Lambda) \text{ such that } g(\psi(X), JY) \\ &= g(\psi(Y), JX),\ \forall X, Y \in T_x\Lambda\}, \end{aligned}$$

so one gets immediately the thesis. □

The tangent application to the Gauss map is related to the second fundamental form as shown in the following.

Lemma 4. *The tangent map to $G(j)$ in a point $x \in \Lambda$ can be identified with the second fundamental form σ, thought of as an application with values in $T^*\Lambda \otimes N\Lambda$; more exactly σ takes values in the subspace $G(j)^*(VT(\mathrm{GrL}))$ of $T^*\Lambda \otimes N\Lambda$, in the sense that it satisfies $g(\sigma(X, Y), JZ) = g(\sigma(X, Z), JY)$.*

Proof. First of all, the identity $g(\sigma(X, Y), JZ) = g(\sigma(X, Z), JY)$ is a consequence of the fact that Lagrangian submanifolds of Kähler manifolds are always anti-invariant (also called totally real) submanifolds of top dimension (see [19, p. 35]). Hence, always by result of [19, p. 43], we have the desired relation. Finally, the fact that the tangent map to the Gauss map can be identified with the second fundamental form, via the action of the almost complex structure J and the metric g, is a classically known result which can be found, for example, in [5, p. 196]. □

Observe that by Lemmas 3 and 4, the second fundamental form $\sigma(X,.)$, considered as a map taking values in $T^*\Lambda \otimes N\Lambda$ is an element of $G(j)^*(VT(\mathrm{GrL}))$. Let us summarize the situation in the following diagram:

$$\begin{array}{ccccc} T\Lambda & \overset{G(j)_*}{\to} & G(j)^*(VT(\mathrm{GrL})) & \subset T^*\Lambda \otimes N\Lambda & VT(\mathrm{GrL}): \\ \downarrow & & & & \downarrow \\ \Lambda & \overset{G(j)}{\to} & G(j)(\Lambda) & \hookrightarrow & \mathrm{GrL}(X)_\Lambda \end{array}$$

Denote again with $\tilde{J}$ the restriction of $\tilde{J}$ to the bundle $G(j)^*(VT^*(\mathrm{GrL}))$. By the previous diagram we can pull-back $\tilde{J}$ to a closed 1-form on Λ via $G(j)^*$

$$(G(j)^*(\tilde{J}))(X) = \tilde{J}(G(j)_*(X)) = \tilde{J}(\sigma(X,.)) \quad \forall X \in \Gamma(T\Lambda), \tag{2}$$

where the last equality in Eq. (2) is due to Lemma 4 and the pairing between $\tilde{J}$ and $\sigma(X,.)$ is induced by the natural pairing between $G(j)^*(VT^*(\mathrm{GrL}))$ and $G(j)^*(VT(\mathrm{GrL}))$, respectively.

Proof of the theorem. First of all, notice that the Maslov map $\mathcal{M} : \Lambda \to S^1$ can be decomposed as $\mathcal{M} = \mathrm{Det}^2 \circ G(j)$, as is immediate to see. Then $\mu_\Lambda := \mathcal{M}^*(\alpha) = G(j)^* \circ (\mathrm{Det}^2)^*(\alpha)$ and so $\mu_\Lambda = (1/\pi)G(j)^*(\tilde{J})$ by Lemma 2. Now $\tilde{J} = e_l \otimes -Jf^l$ and $\sigma(X,.)$ can be represented as $\Gamma(T^*\Lambda \otimes N\Lambda) \ni \sigma(X,.) = \sigma_i^k(X) f^i \otimes Je_k$. In this way we have that for all $X \in \Gamma(T\Lambda)$,

$$\begin{aligned}(G(j)^*(\tilde{J}))(X) &= (e_l \otimes -Jf^l)(\sigma_i^k(X) f^i \otimes Je_k) = \sigma_i^i(X) = \sum_i g(\sigma(X, e_i), Je_i) \\ &= \sum_i g(\sigma(e_i, e_i), JX) = \quad (\text{by Lemma 4}) = g(H, JX) = \omega(H, X) \\ &= i_H\omega(X).\end{aligned}$$

Hence, one gets the result

$$\mu_\Lambda = G(j)^*\left(\frac{1}{\pi}\tilde{J}\right) = \frac{1}{\pi} i_H\omega \quad \in H^1(\Lambda, \mathbb{Z}). \qquad \square \tag{3}$$

By the result of the theorem, one can give the following.

Definition. Let $\Lambda \hookrightarrow X$ be a Lagrangian embedding in a Calabi–Yau X, then the Maslov index m of a closed loop γ on Λ is given by

$$m(\gamma) := \frac{1}{\pi}\int_\gamma i_H\omega \quad \in \mathbb{Z}.$$

4. Conclusions

Calabi–Yau manifolds have received great attention as target spaces for superstring compactifications. Moreover their Lagrangian and special Lagrangian submanifolds are now considered as the cornerstones for understanding the mirror symmetry phenomenon between pairs of Calabi–Yau spaces, both from a categorical point of view [12], and from a physical–geometrical standpoint [15]. Let us recall that special Lagrangian submanifolds Λ of a Calabi–Yau X are exactly what are called BPS states or supersymmetric cycles in the physical literature; on the other hand, it is known that special Lagrangian submanifolds are nothing else other than *minimal* Lagrangian submanifolds (compare [10 p. 96], where this is proved for special Lagrangian submanifolds of $\mathbb{C}^n$). From our result it turns out that the Maslov class of special Lagrangian submanifolds is identically vanishing; on the other hand, this can be seen just by considering the Grassmannian of special Lagrangian planes, which turns out to be diffeomorphic to $\mathrm{SU}(n)/\mathrm{SO}(n)$, hence simply connected (notice that the Grassmannian of special Lagrangian planes is isomorphic to the fibre in the fibration $\det^2 : \mathrm{GrL}(\mathbb{C}^n) \to S^1$). It is then clear that the Maslov index is identically vanishing for all special Lagrangian submanifolds Λ of a Calabi–Yau X. We believe that this simple observation can enhance our understanding of the structure of the A^∞-Fukaya category, whenever its objects are restricted to minimal Lagrangian submanifolds (see [7] for a definition of A^∞ category, and [12] for its application in the study of mirror symmetry). Indeed, this is a key point for the proof of homological mirror symmetry for K3 surfaces, for which we refer to [3].

The Maslov class so far constructed does not depend on the choice of a canonical projection, from which one could determine the singular locus (as it usually happens when one considers Lagrangian embedding in cotangent bundles over an arbitrary Riemannian manifold). However, it is still possible to determine, rather than the singular locus, the *homology class* $[Z] \in H_{n-1}(\Lambda, \mathbb{Z})$ of a “singular locus”, just considering the Poincaré dual to $[\mu_\Lambda]$, and setting $[Z] := \mathrm{Pd}([\mu_\Lambda])$ (Pd stands for Poincaré duality). We have said “a singular locus”, because Z is not determined at all uniquely, but only up to its homology class; in spite of this one could take as singular locus any representative of $[Z]$. So it makes sense to speak of a singular locus, even if there is no projection to refer it.

It is clear that it is not possible to extend our definition of Maslov class for Lagrangian embedding in arbitrary symplectic manifolds; even the construction of Fukaya (which is specifically designed for Maslov index of closed loops only on BS orbits) needs several assumption such that the ambient manifold admits a “prequantum bundle” and so on. We are thus tempted to suggest the following alternative description: we would like to define the Maslov class for a Lagrangian embedding in *any* symplectic manifold (X, ω) via the mean curvature representation $i_H\omega$. Two problems arise following this approach. First of all, to define the mean curvature vector field H it is necessary to fix a Riemannian metric on X; as it is well known, on any symplectic manifold one has lots of Riemannian metrics $g_J(X, Y) := \omega(X, JY)$, constructed using the given symplectic form ω and choosing an ω-compatible almost complex structure J (recall that the set of ω-compatible almost complex structures on a given symplectic manifold is always nonempty and contractible, see [8]). What is the “right” choice for g_J?

Once we have fixed the right metric, the second problem is related to the closure of the 1-form $i_H\omega$, considered as a form on Λ; indeed there is no reason, a priori, for which $i_H\omega$ has to be closed. We are thus led to the following.

Conjecture. Having fixed the Lagrangian embedding $j : \Lambda \hookrightarrow X$ on any symplectic manifold (X, ω), there exists at least one Riemannian metric g_J built up from an ω-compatible almost complex structure J, such that the 1-form $i_H\omega$ considered as a form on Λ is closed. Multiplying the corresponding cohomological class $[i_H\omega]$ for a suitable constant in such a way that it is integer valued, we call this class the *Maslov–Morvan class* of the Lagrangian submanifold Λ.

It does not seem possible to give an interpretation of this conjectured Maslov–Morvan class via the universal Maslov class, as we have done for Calabi–Yau manifolds, since, in general, we have no control on the holonomy of g_J.

Clearly, the study of the relations between the conjectured class $[i_H\omega]$ and the ordinary Maslov class for a Lagrangian embedding in cotangent bundles (via Morse families) deserves further effort and is left for future investigations.

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References

[1] V.I. Arnol'd, Characteristic classes entering in quantization conditions, Funct. Anal. Appl. 1 (1967) 1–13.
[2] V.I. Arnol'd, A.B. Givental', Symplectic geometry, in: Dynamical Systems, Vol. 4, Springer, Berlin, 1990.
[3] C. Bartocci, U. Bruzzo, S. Sanguinetti, Categorial mirror symmetry for K3 surfaces, Commun. Math. Phys. 206 (1999) 265–272.
[4] A.L. Besse, Einstein manifolds, Modern Surveys in Mathematics, Springer, Berlin, 1987.
[5] R.L. Bishop, R.J. Crittenden, Geometry of Manifolds, Academic Press, New York, 1964.
[6] R. Bott, L.W. Tu, Differential Forms in Algebraic Topology, Springer, Berlin, 1982.
[7] K. Fukaya, Morse homotopy, A^∞-category and Floer homologies, Preprint, MSRI, Berkeley, 1993.
[8] M. Gromov, Pseudo-holomorphic curves in symplectic manifolds, Invent. Math. 82 (1985) 307–347.
[9] V. Guillemin, S. Sternberg, Geometric Asymptotics, Mathematical Surveys, Vol. 14, American Mathematical Society, Providence, RI, 1977.
[10] R. Harvey, H.B. Lawson, Calibrated geometries, Acta Math. 148 (1982) 47–157.
[11] M.V. Karashev, V.P. Maslov, Nonlinear Poisson brackets, Geometry and Quantization, American Mathematical Society, Providence, RI, 1993.
[12] M. Kontsevich, Homological Algebra of Mirror Symmetry, alg-geom/9411018.
[13] V.P. Maslov, Perturbation Theory and Asymptotic Methods, Dunod, Paris, 1972.
[14] J.-M. Morvan, Classe de Maslov d' une immersion lagrangienne et minimalité, C.R. Acad. Sc. Paris, Série I, 292, 1981.
[15] A. Strominger, S.-T. Yau, E. Zaslow, Mirror symmetry is T-duality, Nucl. Phys. B 479 (1996) 243–259.
[16] V.V. Trofimov, Generalized Maslov classes on the path space of a symplectic manifold, Proceedings of the Steklov Institute of Mathematics, Issue 4, 1995.
[17] C. Voisin, Symétrie Miroir, SMF, Panoramas et Synthéses, 1996.
[18] A. Weinstein, Lectures on Symplectic Manifolds, Regional Conference Series in Mathematics, American Mathematical Society, Providence, RI, 1977.
[19] K. Yano, M. Kon, Anti-invariant submanifolds, Lecture Notes in Pure and Applied Mathematics, Vol. 21, Marcel Dekker, New York, 1976.